

Bias and dessins

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1 Introduction

1.1. Abstract. Grothendieck's theory of dessins provides a bridge between algebraic numbers and combinatorics. This paper adds a new concept, called *bias*, to the bridge. This produces: (i) from a biased plane tree the construction of a sequence of algebraic numbers, and (ii) a Galois invariant lattice structure on the set of biased dessins. Bias brings these benefits by (i) using individual polynomials instead of equivalence classes of polynomials, and (ii) applying properties of covering spaces and the fundamental group. The new features give new opportunities.

At the 2014 SIGMAP conference the author spoke [1] on *The decorated lattice of biased dessins*. This decorated lattice \mathcal{L} is combinatorially defined, and its automorphism group contains the absolute Galois group Γ , perhaps as an index 6 subgroup.

This paper defines new families of invariants of dessins, although they require further work to be understood and useful. For this, \mathcal{L} is vital. This paper relies on the existing, unbiased, theory. Also, it only sketches the construction of \mathcal{L} . In [2, 3] the author will remove this dependency, develop the biased theory further, with a focus on Γ , and make the theory more accessible.

1.2. Advice to the reader. This paper is a compromise. Either directly or in the background it involves algebraic numbers, algebraic geometry, analysis, combinatorics, Galois theory and topology. What assumptions to make of the reader? For example, the Galois invariance of the lattice structure (Theorem 3.14) will be obvious to some readers, and mysterious to others. The paper assumes only what is required to achieve its limited goal.

This goal is to show that the addition of bias greatly improves the existing theory of dessins. Central to dessins is the bijection given by the bridge between algebraic numbers and combinatorics. Theorem 2.19 gives a bridge which carries bias. Given the stated analogous unbiased result, its proof should be accessible to all readers. This gives (see §2.6) many new Galois invariants for biased Shabat polynomials.

The join operation on biased dessins is new. Its combinatorial description (see Definition 3.6) is simple and attractive. It is also Galois invariant, which we prove elsewhere [3]. It gives a powerful method (see §3.6) of producing new Galois invariants of biased dessins from old. In §3.7 to §3.10 we motivate and sketch the definition of the decorated lattice \mathcal{L} of biased dessins. This provides the ground for the definition of further new Galois invariants of dessins.

Even when studying unbiased objects, use of bias is a great help (see §4.3). The author will in [2, 3] present the theory of dessins anew, but with bias introduced from the very beginning, rather than as an afterthought (the present paper's approach). Further, the focus will be on the absolute Galois group, and on making the theory more widely accessible. Until then, there is the present paper, with its limited goal.

In the rest of this section we give the basic concepts on which most of this paper is based. Sections §3.7–3.10 require further background.

1.3. Algebraic numbers. We let $\mathbb{Q} \subset \mathbb{C}$ denote the rational and complex numbers. Let $f(z)$ be a polynomial in z , with rational coefficients. If $f(u) = 0$ we say that u is an *algebraic number* (provided $u \in \mathbb{C}$ and f is not constant). The algebraic numbers form a field, $\overline{\mathbb{Q}}$, lying between \mathbb{Q} and \mathbb{C} .

We let $\mathbb{Q}[z] \subset \overline{\mathbb{Q}}[z] \subset \mathbb{C}[z]$ denote polynomials with coefficients in $\mathbb{Q} \subset \overline{\mathbb{Q}} \subset \mathbb{C}$ respectively. By the fundamental theorem of algebra (a topological result proved by Gauss), the field \mathbb{C} is *algebraically closed*. In other words, any $f \in \mathbb{C}[z]$ has $n = \deg f$ roots, when counted with multiplicity. The same is true of $\overline{\mathbb{Q}}$, but this is an algebraic result.

Definition 1.1. *The absolute Galois group Γ consists of all field automorphisms of $\overline{\mathbb{Q}}$.*

We need some simple results about Γ . We use $u \mapsto \tilde{u}$ to denote an element of Γ . Always, $\tilde{u} = u$ for $u \in \mathbb{Q}$. By acting on coefficients this induces a map $f \mapsto \tilde{f}$ on $\overline{\mathbb{Q}}[z]$. Because $u \mapsto \tilde{u}$ is a field automorphism, it follows that $\tilde{f}(\tilde{u}) = \tilde{v}$, where $v = f(u)$. Similarly, for derivatives. The expression \tilde{f}' can be evaluated in two ways: first apply $u \mapsto \tilde{u}$ and then the derivative, or vice versa. Both give the same result, which we denote by \tilde{f}' .

The inclusion $\overline{\mathbb{Q}} \subset \mathbb{C}$ induces a topology on $\overline{\mathbb{Q}}$. Note that $u \mapsto \tilde{u}$ in Γ is not continuous for this topology, unless it is either the identity map $u \mapsto u$ or complex conjugation $u \mapsto \bar{u}$.

1.4. Galois invariants and the minimal polynomial. We are interested in Galois invariants of trees and dessins, and we would like a complete set of such invariants. The minimal polynomial is a basic example of a complete Galois invariant.

Let $a \in \overline{\mathbb{Q}}$ be an algebraic number. Of all non-zero $f \in \mathbb{Q}[z]$ such that $f(a) = 0$ there is only one that (i) has least degree, and (ii) has top-degree coefficient 1. This is called the *minimal polynomial* $g_a(z) \in \mathbb{Q}[z]$ of a .

Suppose $b = \tilde{a}$ for some $u \mapsto \tilde{u}$ in Γ . It is easily proved that $g_a = g_b$. Put another way, the minimal polynomial g_a is a *Galois invariant* of $a \in \overline{\mathbb{Q}}$. Now suppose $g_a = g_b$. Does it follow that there is a $u \mapsto \tilde{u}$ in Γ , such that $b = \tilde{a}$. If so, then we say that the minimal polynomial is a *complete* Galois invariant. For use in Proposition 2.9, note that $f \mapsto f'$ for $f \in \overline{\mathbb{Q}}[z]$ is an example of something that is *Galois covariant*. Equivalently, the truth of the statement “the derivative of f is g ” is Galois invariant (for $f, g \in \overline{\mathbb{Q}}[z]$).

Proposition 1.2. *Suppose $a \in \overline{\mathbb{Q}}$. Then the minimal polynomial $g_a(z) \in \mathbb{Q}[z]$ is a complete Galois invariant of a .*

The completeness of the minimal polynomial is a fundamental property of the absolute Galois group. It states that certain incomplete automorphisms of $\overline{\mathbb{Q}}$ can be indefinitely extended.

1.5. Critical points and values. Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is a polynomial function. If $f'(u) = 0$ for $u \in \mathbb{C}$ we say that u is a *critical point* of f , and that $v = f(u)$ is a *critical value*. For each $u \in \mathbb{C}$ let $v = f(u)$ and consider the polynomial equation $f(z) - v = 0$. Writing

$$f(z) = v + a_1(z - u) + a_2(z - u)^2 + \dots + a_n(z - u)^n$$

we see that $z = u$ is a simple root of $f(z) - v = 0$ if and only if $f'(u) \neq 0$.

Thus, provided $v \in \mathbb{C}$ is not a critical value of $f : \mathbb{C} \rightarrow \mathbb{C}$, the fibre $f^{-1}(v)$ of f consists of n distinct points, at each of which f' is non-zero. Using the language of topology (§3.7) we have that $f : \mathbb{C} \rightarrow \mathbb{C}$ is a covering map away from the critical values.

1.6. Bipartite plane trees. The reader will need enough combinatorics to understand the following result, which we will explain. Figure 1b shows a bipartite plane tree. (By the way, Figure 1a is a biased plane tree.)

Proposition 1.3. *A bipartite plane tree is equivalent to an irreducible pair of permutations such that $\alpha\beta$ has at most one orbit.*

First, a word about equality. We will say that two combinatorial objects are equal if the one can be transformed into the other by relabelling. Thus, we are implicitly talking about equivalence classes of labelled objects. For example, any two graphs that have only one vertex (and hence no edges) are equal, i.e. belong to the same equivalence class.

In this paper: (1) A *graph* G is a set $V = V_G$ of *vertices* together with the *edges* $E = E_G$, a set of unordered pairs of vertices. (2) All graphs, trees and dessins will have a finite number of vertices and edges. (3) A *path* is a sequence of edges of the form $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_n, v_{n+1}\}$ such that the v_i are distinct. (4) A *tree* is a graph where there is exactly one path between any two distinct vertices. This condition allows the no-vertex and one-vertex graphs as trees. (5) A *bipartite* graph is one where (i) V_G is partitioned into two subsets, the *black* and *white vertices*, and (ii) each edge has a black vertex and a white vertex. (6) For consistency with $n = \deg f$, we let $\deg X$ denote the number of edges in X , for X a graph, tree or (to be defined later) dessin.

In addition: (7) The plane will always be \mathbb{C} , with its usual counter-clockwise orientation. (8) A *plane graph* will be a graph that is drawn on the plane, with edges intersecting only at the endpoints. (9) Thus, a bipartite plane tree is (i) a plane graph, (ii) with exactly one path between any two vertices, and (iii) an alternate black and white labelling of the vertices.

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Figure 1: (a) A biased plane tree. (b) The corresponding bipartite plane tree.

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Figure 2: (a) The permutation α . (b) The permutation β . (c) The permutation $\alpha\beta$.

1.7. Pairs of permutations. First, a word about the figures. Figure 1a shows a biased plane tree T , and Figure 1b shows the resulting bipartite plane tree T' (which has at least one edge). Figure 2a shows the permutation α on the edges of T' (and hence T), and Figure 2b the

permutation β . Finally, Figure 2c show the permutation $\alpha\beta$ on the edges of T' . The key point of Figure 2c is that $\alpha\beta$ is a counterclockwise ‘two-step walk around’ T' , which visits each side of each edge exactly once.

Consider the edges in Figure 1b. Each edge e has a black vertex. Rotating counterclockwise around that vertex we come to another (or possibly the same) edge e_1 . We will write $e_1 = \alpha(e)$. Similarly, we define $\beta(e)$ by rotating counterclockwise around the white vertex of e . Figure 2 parts (a) and (b) show α and β respectively. Clearly, each bipartite plane tree T determines a pair of permutations (α, β) on the edge set $E = E_T$ of T .

Here’s how the process can be reversed: (1) A *permutation* is a bijection $\alpha : E \rightarrow E$ from a set to itself. (2) A *pair of permutations* P is an ordered pair (α_P, β_P) of permutations of the same set $E = E_P$. We call E the *edges* of P . We require E to be a finite set. (3) We let V_b denote the α -orbits in E , and V_w the β -orbits. (4) We let V be the disjoint union of V_b and V_w . We may need to relabel V_b or V_w , for example when E has only one element. (5) Let E' be the pairs $\{v_b, v_w\}$ where v_b and v_w are orbits of the same edge $e \in E$. (6) We can, and will, identify E and E' . By construction, there is at most one edge between two vertices.

This produces, from any pair of permutations P , (i) a bipartite graph G_P , together with (ii) at each v of G_P a cyclic order on the edges lying on that v . Conversely, such data determines a pair of permutations. When is G_P connected? The reader is asked to check:

Notation 1.4. $\langle \alpha, \beta \rangle$ is the group generated by α and β .

Definition 1.5. A pair of permutations P is **irreducible** if E_P is either empty or an orbit of $\langle \alpha_P, \beta_P \rangle$.

Proposition 1.6. Let P be a pair of permutations. The graph G_P is connected if and only if P is irreducible.

We now return to the proof of Proposition 1.3. Let T be a bipartite plane tree, with $\deg T \geq 1$, and P the associated pair of permutations. We have seen that P is irreducible and that $\alpha\beta$ has a single orbit on the edges of P . Now cut the plane along T and, using rubber sheet geometry, deform the cut plane until: (i) it is a disc that is removed, and (ii) the boundary circle is divided into $2n$ arcs.

Because $\deg T \geq 1$, it has a vertex v that lies on only one edge e . Suppose v is black. It follows that $\alpha(e) = e$. Removing e from T glues back together two adjacent edges of the boundary circle. The result now follows if we can prove: (i) the hypothesis on $\alpha\beta$ implies that we can always find such an edge, and (ii) after removal of this edge the new $\alpha\beta$ still satisfies the hypothesis. This will be done in [2], or the reader can treat it as an exercise.

2 Shabat polynomials and plane trees

2.1. Unbiased Shabat polynomials. We start with a summary of already known definitions and results. What others have called a *Shabat polynomial* we call, for clarity, an *unbiased Shabat polynomial*. The same applies to *dessins* and *unbiased dessins*.

Definition 2.1. An **unbiased Shabat polynomial** is a non-constant polynomial function $f : \mathbb{C} \rightarrow \mathbb{C}$ together with an ordered pair (v_b, v_w) of distinct points in \mathbb{C} , such that if $f'(u) = 0$ then $f(u) \in \{v_b, v_w\}$.

We call v_b and v_w the *black* and *white vertices* respectively, and throughout will write $v_0 = (v_b + v_w)/2$ for the midpoint of the line segment or *edge* $[v_b, v_w]$ that joins them. Note that v_b and v_w need not be critical values. For example, $z \mapsto z$ is unbiased Shabat, for any distinct v_b and v_w .

Definition 2.2. A **change of coordinates** (on \mathbb{C}) is a map $\psi : \mathbb{C} \rightarrow \mathbb{C}$ of the form $\psi(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$.

Notation 2.3. \mathcal{S}'_n consists of all unbiased Shabat polynomials of degree n , modulo change of coordinates on both domain and range. We write $\mathcal{S}' = \bigcup \mathcal{S}'_n$.

Thus, each element s of \mathcal{S}' is an *equivalence class* of unbiased Shabat polynomials. This is why we need bias. We use bias to (i) reduce s to a finite set of representatives, and then (ii) choose one of the representatives. A polynomial $f \in \overline{\mathbb{Q}}[z]$ is much closer to algebraic numbers than an unbiased Shabat equivalence class. This is a great help (see §2.3).

The reader is asked to check the following. (1) Change of coordinates preserves the degree of f . (2) Composition of functions induces a group structure on the set of changes of coordinates. (3) If f is unbiased Shabat then so is $f \circ \psi$, with the same vertex pair (v_b, v_w) . (4) Similarly, $\psi \circ f$ is also unbiased Shabat, but with the pair $(\psi(v_b), \psi(v_w))$. (5) Given unbiased Shabat f there is a unique ψ such that (v_b, v_w) becomes $(-1, +1)$ when we apply ψ to produce $\psi \circ f$.

Definition 2.4. \mathcal{T}'_n consists of all non-empty bipartite plane trees with n edges, and $\mathcal{T}' = \bigcup \mathcal{T}'_n$.

As usual, \mathcal{T}' is up to relabelling combinatorial equivalence. The next result is Grothendieck's bridge. For a proof see [5], [8] or [2].

Theorem 2.5. The map $f \mapsto T_f = f^{-1}([v_b, v_w])$ induces a bijection between \mathcal{S}'_n and \mathcal{T}'_n .

T_f is a combinatorial, and hence topological, description of f . This is because T_f can be used as the data for a gluing construction, via covering spaces (see §3.7), that gives a map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ that is topologically equivalent to $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. For details see [5] or [2].

The theorem states that (i) change of coordinates does not change the combinatorial structure of T_f (this is left to the reader), (ii) T_f is a bipartite plane tree, and (iii) we can reconstruct f from T_f , up to change of coordinates. Put another way, topology determines geometry. In §2.5 we add bias to both f and T_f . We do this so that f to be reconstructed exactly, without the change of coordinates indeterminacy.

The following are key for the usefulness of the bridge. For a proof see [5], [8] or [2].

Lemma 2.6. Each equivalence class $s = [f]$ in \mathcal{S}' has at least one element f_1 that lies in $\overline{\mathbb{Q}}[z]$.

Theorem 2.7. Γ acts on \mathcal{S}'_n , and its action on \mathcal{S}' is faithful.

2.2. Goals. The bijection between \mathcal{S}' and \mathcal{T}' produces an action of the absolute Galois group Γ on \mathcal{T}'_n . Understanding this action combinatorially, without going over the bridge into algebraic numbers, would help us understand Γ . Some first steps are to find Galois invariants of \mathcal{T}' , and to understand the decomposition of \mathcal{T}'_n into orbits.

The main goal is understanding Γ . For us biased and unbiased objects are a means to an end. The main idea of this paper is that the goal is better reached by using biased objects.

2.3. Choosing f in $s \in \mathcal{S}'_n$. We want Galois invariants of $s \in \mathcal{S}'_n$. If each element of \mathcal{S}'_n were a polynomial $f \in \overline{\mathbb{Q}}[z]$ then the minimal polynomials $g_i(z) \in \mathbb{Q}[z]$ of the coefficients a_i of f would be Galois invariants of f and hence of T_f . But each element s of \mathcal{S}'_n is an equivalence class of unbiased Shabat polynomials, not a single such polynomial.

If we could in an Galois invariant way choose an f in s , then we could use that f instead of s . This seems not to be possible, but we can come close enough. We can define a non-empty finite subset of s , in a Galois invariant manner (see also §4.4). Choosing an element from this subset we call the process of *biasing* f (in its equivalence class).

Suppose f is unbiased Shabat. Let f_1 be $\psi \circ f \circ \eta$, for changes of coordinates ψ and η . We want to choose ψ and η so f_1 is fixed, up to a finite choice. Already, the reader has checked that there is a unique ψ such that $(-1, 1)$ is the black-white vertex pair associated with f_1 . The uniqueness is important. We now need a condition that determines η .

Let f_0 be $\psi \circ f$. It has vertex pair $(-1, 1)$. Now consider the equation $f_0(z) = 0$. Counted with multiplicity, this has n roots. Let $u \in \mathbb{C}$ be one of them. If $f'_0(u) = 0$ then, by the Shabat condition, $f_0(u) \in \{-1, +1\}$. Thus, $f'_0(u) \neq 0$ and $f_0(z) = 0$ has exactly n distinct roots.

Recall that $f_1 = f_0 \circ \eta$. Assume that $f_0(u) = 0$. This is the finite choice. The change of coordinates η has two degrees of freedom. If $\eta(z) = az + u$ then $f_1(0) = f_0(u) = 0$. Assume η has this form. This leaves a to be determined. Now consider $f'_1(0)$. By the chain rule we have $f'_1(0) = af'_0(u)$. We have just seen that $f'_0(u) \neq 0$ and so we can write $a = 1/f'_0(u)$ to give $f'_1(0) = 1$. The bias is a choice of one of the n roots of $f_0(z) = 0$, or equivalently $f(z) = v_0$, where as usual $v_0 = (v_b + v_w)/2$.

2.4. Applying the choosing process. Here we summarize the §2.3, and prepare for bias. The previous discussion shows:

Proposition 2.8. *Suppose f is unbiased Shabat, with vertex pair (v_b, v_w) . Suppose also that $u \in \mathbb{C}$ is a root of $f(z) = v_0$. Then there is a unique pair ψ, η of changes of coordinates such that (i) $\psi(v_b) = -1$ and $\psi(v_w) = +1$, (ii) $\eta(0) = u$, and (iii) $(\psi \circ f \circ \eta)'(0) = 1$.*

Note that ψ is affine linear, so $\psi(v_0) = ((-1) + (+1))/2 = 0$ and thus $(\psi \circ f \circ \eta)(0) = 0$.

Proposition 2.9. *Let f and u be as above, and let f_1 be the resulting $\psi \circ f \circ \eta$. Then:*

1. f_1 is biased Shabat, as in Definition 2.11 below.
2. If $f \in \overline{\mathbb{Q}}[z]$ then f_1 is also in $\overline{\mathbb{Q}}[z]$.
3. Applied to \tilde{f} and \tilde{u} the construction yields \tilde{r} , where $r = f_1$. In other words, the construction is Galois covariant.

Proof. Parts (1) and (3) are left to the reader. *Biased Shabat* is defined as it is, to make (1) true. Part (3) is needed for the proof of Theorem 2.19. Its proof is purely formal.

The proof of (2) has a tricky special case. Suppose $f \in \overline{\mathbb{Q}}[z]$. By Lemma 2.10 below the critical values of f lie in $\overline{\mathbb{Q}}$. If f has two critical values then $v_b, v_w \in \overline{\mathbb{Q}}$. This is enough to ensure $\psi \in \overline{\mathbb{Q}}[z]$, as $\overline{\mathbb{Q}}$ is a field. Similarly, $u \in \overline{\mathbb{Q}}$ as $f(u) = v_0$ and $\overline{\mathbb{Q}}$ is algebraically closed, and thus $\eta \in \overline{\mathbb{Q}}[z]$. As $f, \psi, \eta \in \overline{\mathbb{Q}}[z]$ it follows that $f_1 = \psi \circ f \circ \eta \in \overline{\mathbb{Q}}[z]$.

We now have to deal with the special cases. The first is easy. If f has no critical values then it is a change of coordinates. We ask the reader to check that the process results in $f_1(z) = z$.

Now assume f has exactly one critical value, say v_b . This requires a trick. Consider T_f . By Theorem 2.5, it is a plane tree. By assumption, the white vertices are not critical points, and so lie on only one edge. Thus, T_f is an n -pointed star, with a black vertex at the centre. But $p(z) = z^n$ with $(0, 1)$ also gives T_f and so, again by Theorem 2.5, some change of coordinates will take f to p . We are now out of the special case, and the previous argument produces a $p_1 \in \overline{\mathbb{Q}}[z]$. By uniqueness of the change of coordinates (see Proposition 2.8), we have $f_1 = p_1$. The author does not see how to avoid using Theorem 2.5, or something similar. \square

Lemma 2.10. *Suppose $f \in \overline{\mathbb{Q}}[z]$. Then the critical values of f lie in $\overline{\mathbb{Q}}$.*

Proof. Suppose $\deg f' \geq 1$, and $f'(u) = 0$. It follows that $u \in \overline{\mathbb{Q}}$ (as $\overline{\mathbb{Q}}$ is algebraically closed) and then $v = f(u) \in \overline{\mathbb{Q}}$ (as $\overline{\mathbb{Q}}$ is a field). The remaining case, $f(z)$ constant, is trivial. \square

2.5. Biased Shabat polynomials. Here we add bias to the definitions, and thereby remove equivalence classes from the polynomial end of the bridge. This will give new Galois invariants.

Definition 2.11. *A biased Shabat polynomial is a polynomial function $f : \mathbb{C} \rightarrow \mathbb{C}$ such that (i) if $f'(u) = 0$ then $f(u) \in \{-1, +1\}$, (ii) $f(0) = 0$, and (iii) $f'(0) = 1$.*

Notation 2.12. \mathcal{S}_n is all biased Shabat polynomials of degree n , and $\mathcal{S} = \bigcup \mathcal{S}_n$.

Proposition 2.13. *If f is biased Shabat then $f \in \overline{\mathbb{Q}}[z]$.*

Proof. This follows from the unbiased result. Think of f as unbiased Shabat. By Theorem 2.5 there is a change of coordinates (ψ, η) that produces from f an unbiased $\psi \circ f \circ \eta = f_1 \in \overline{\mathbb{Q}}[z]$. Now bias f_1 , choosing $\eta^{-1}(0)$ as the solution u of $f_1(z) = v_0$. By Proposition 2.9 the result f_2 lies in $\overline{\mathbb{Q}}[z]$. By Proposition 2.8 the change of coordinates that does this is unique. So it must be (ψ^{-1}, η^{-1}) and thus $f = f_2$ lies in $\overline{\mathbb{Q}}[z]$. \square

Corollary 2.14. Γ acts on \mathcal{S}_n , by acting on the coefficients.

Proof. This is because the biased Shabat conditions are Galois invariant. For example, if $f'(u) = 0$ then $\tilde{f}'(\tilde{u}) = \tilde{0} = 0$, and vice versa. Similarly, $f(u) = -1$ if and only if $\tilde{f}(\tilde{u}) = -1$. The same applies to $f(u) = +1$, $f(0) = 0$ and $f'(0) = 1$. \square

Corollary 2.15. *The action of Γ on \mathcal{S} is faithful.*

Proof. The forget-bias map $\mathcal{S} \rightarrow \mathcal{S}'$ is surjective, and consistent with the Galois action. The Galois action is faithful on \mathcal{S}' , by Theorem 2.7. \square

2.6. Galois invariants. Recall (see Proposition 1.2) that each $a \in \overline{\mathbb{Q}}$ has a minimal polynomial $g_a(z) \in \mathbb{Q}[z]$, and that g_a is a complete Galois invariant for a . Let $f(z) = a_0 + a_1z + \dots + a_nz^n$ be a polynomial in $\overline{\mathbb{Q}}[z]$. Clearly, the sequence $g_i(z) \in \mathbb{Q}[z]$ of the minimal polynomials of the coefficients a_i is a Galois invariant of f . Thus we obtain many Galois invariants of biased Shabat polynomials. Of course, for f biased Shabat $a_0 = 0$ and $a_1 = 1$, and so g_0 and g_1 are constant on \mathcal{S} .

On $\overline{\mathbb{Q}}[z]$, the sequence of minimal polynomials is not a complete Galois invariant. For example, all coefficients of $f_-(z) = \sqrt{2}(1 - z)$ and $f_+(z) = \sqrt{2}(1 + z)$ have $g(z) = z^2 - 2$ as their minimal polynomial. But $f_-(1) = 0 \in \mathbb{Q}$ while $f_+(1) = 2\sqrt{2} \notin \mathbb{Q}$. The author suspects that there are distinct $f_1, f_2 \in \mathcal{S}_n$ with $g_{1,r}(z) = g_{2,r}(z)$ for all $r \leq n$.

2.7. Biased plane trees. Recall that unbiased Shabat polynomials correspond to bipartite plane trees. For biased polynomials, we want a similar corresponding definition. Let f be biased Shabat. Consider $T_f = f^{-1}([-1, 1])$. By forgetting the bias we see, as before, that T_f is a plane tree with a bipartite colouring of the vertices. Because $f(0) = 0 \in [-1, 1]$, we have $0 \in T_f$. In fact, each of the n edges has an interior point c such that $f(c) = 0$, and so 0 lies on a single edge e_f of T_f .

Thus, even in the unbiased case, the choice of a root of $f(z) = v_0$ is equivalent to the choice of an edge in T_f . If f is biased then $f(0) = 0$ is the chosen root. This gives rise to:

Definition 2.16. *A biased plane tree T is a bipartite plane tree with a chosen edge e_T .*

Now draw the tree, and an arrow, black vertex to white, on the chosen edge. This, by itself, is enough to determine the colour of all other vertices of the tree (see Figure 1), and we still have a chosen edge. Thus, the previous definition is equivalent to:

Definition 2.17. *A biased plane tree is a plane tree with an arrow (the bias) along one edge.*

We can now state the biased analogue of Theorem 2.5.

Notation 2.18. \mathcal{T}_n is all biased plane trees with n edges, and $\mathcal{T} = \bigcup \mathcal{T}_n$.

Theorem 2.19. *The map $f \mapsto T_f = f^{-1}([-1, -1])$ induces a bijection between \mathcal{S}_n and \mathcal{T}_n .*

Proof. Think of a biased f as an unbiased f , together with a root c of the equation $f(z) = v_0$. Now use the bijection between \mathcal{S}'_n and \mathcal{T}'_n provided in Theorem 2.5. We can use c to select an edge on T_f , and vice versa. This lifts the bijection to \mathcal{S}_n and \mathcal{T}_n . \square

2.8. Rooted plane trees and Catalan numbers. A biased plane tree is the same as a rooted plane tree, as used in linguistics and computer science for parse and syntax trees, except that a rooted plane tree need not have any edges. Thus, *biased plane tree* is a shorthand for *rooted plane tree with at least one edge*. For us, the black-white alternation of vertices along edges is important, as is the presently mysterious Galois action.

It is well known that the number of rooted plane trees with n edges is the n -th Catalan number. As Γ acts faithfully on \mathcal{T} , it also acts faithfully on any set that is in bijection with rooted plane trees. There are many interesting examples of such [9]. This will be explored further in [2].

3 Dessins

3.1. Overview. In the previous section we introduced bias to solve a geometric problem, namely that unbiased T determines f only up to change of coordinates. In this section we add bias to solve a combinatorial problem, namely that the Cartesian product of two trees is not a tree. To do this we also have to generalise tree to dessin. We use the same concept of bias. This process puts a Galois invariant lattice structure on the set of biased dessins. We can use this (see §3.6) to define new Galois invariants from old.

3.2. Unbiased dessins. Recall (Proposition 1.3) that a bipartite plane tree is equivalent to an irreducible pair $P = (\alpha, \beta)$ of permutations, such that $\alpha\beta$ has at most one orbit. Sets have a Cartesian product, and something similar can be done for pairs of permutations.

Definition 3.1. For pairs of permutations P_1 and P_2 the **product** $P_1 \times P_2$ has edge set $E_1 \times E_2$ and permutations $\alpha((e_1, e_2)) = (\alpha_1(e_1), \alpha_2(e_2))$, and similarly for β .

The product $T = R \times S$ of two pairs of permutations is also a pair of permutations. Even when R and S are irreducible, T may be reducible. For example, $R \times R$ is reducible if R has two or more edges. This is because its diagonal $\{(e, e) | e \in R\}$ is irreducible, but is not the whole of $R \times R$. However, $R \times S$ always decomposes into irreducibles, each of which is an $\langle \alpha, \beta \rangle$ orbit.

We generalise the concept of unbiased plane tree as follows:

Definition 3.2. An **unbiased dessin** is an irreducible pair D of permutations, where D has at least one edge.

Note that each product of unbiased dessins, which may be reducible, has a unique decomposition into unbiased dessins.

Notation 3.3. \mathcal{D}'_n is all unbiased dessins with n edges, and $\mathcal{D}' = \bigcup \mathcal{D}'_n$.

3.3. Biased dessins. We have just seen that the product $T = R \times S$ of two unbiased dessins is sometimes reducible, and so not a dessin. We will choose a component of T as follows:

Definition 3.4. A **biased dessin** D is an irreducible pair of permutations, together with a chosen edge e_D of D .

Notation 3.5. \mathcal{D}_n is all biased dessins with n edges, and $\mathcal{D} = \bigcup \mathcal{D}_n$.

Definition 3.6. The **join** $T = R \vee S$ of two biased dessins is the $\langle \alpha_T, \beta_T \rangle$ orbit of (e_R, e_S) in the product $R \times S$, with chosen edge $e_T = (e_R, e_S)$.

3.4. Morphisms. Suppose R and S are pairs of permutations. A *morphism* $\psi : R \rightarrow S$ is a set map $\psi : E_R \rightarrow E_S$ such that $\psi \circ \alpha_R = \alpha_S \circ \psi$ and similarly for β . We use the same concept for unbiased dessins.

Definition 3.7. A **morphism** $\psi : R \rightarrow S$ of biased dessins is a pair of permutations morphism, call it ψ , such that $\psi(e_R) = e_S$.

Each biased dessin is $\langle \alpha, \beta \rangle$ irreducible, and morphisms respect the chosen edge. From this it easily follows that:

Lemma 3.8. *For any two biased dessins R and S there is at most one morphism $\psi : R \rightarrow S$.*

Notation 3.9. *For biased dessins we write $R \rightarrow S$ if there is a morphism $\psi : R \rightarrow S$.*

Thus we can think of $R \rightarrow S$ either as a boolean relation between R and S , or as the combinatorial structure that makes this relation true. Clearly, $R \rightarrow S$ is a partial order. In [3] we will prove:

Theorem 3.10. *The relation $R \rightarrow S$ gives \mathcal{D} a lattice structure, with join as in Definition 3.6.*

3.5. Marked Belyi pairs. Extending the bijection between \mathcal{S} and \mathcal{T} , there is a concept of marked Belyi pair such that:

Notation 3.11. \mathcal{B}_n is all marked Belyi pairs of degree n , and $\mathcal{B} = \bigcup \mathcal{B}_n$.

Theorem 3.12. Γ acts on \mathcal{B}_n . The action on \mathcal{B} is faithful.

Theorem 3.13. The map $f \mapsto D_f = f^{-1}([-1, -1])$ induces a bijection between \mathcal{B}_n and \mathcal{D}_n .

Theorem 3.14. The lattice structure on \mathcal{B} is Galois invariant under this bijection.

The proof of these results, and the definition of marked Belyi pair, will be given in [3]. The proof can be done, as in Theorem 2.19, by adding bias to the corresponding unbiased result.

3.6. The tower of Galois invariants. We can use the lattice structure on \mathcal{B} to produce new Galois invariants from old. Let $h : \mathcal{B} \rightarrow \mathcal{V}$ be any Galois invariant, such as the degree (number of edges), or the partition triple (see Proposition 3.18). If $R \in \mathcal{B}$ is Galois invariant then so is the function $X \mapsto h(R \vee X)$. Now suppose $S \subset \mathcal{B}$ is a Galois invariant subset. Using formal sums (see below) we have that

$$h_S(X) = \sum_{Y \in S} [h(Y \vee X)] \quad (1)$$

is also Galois invariant. Something similar can be done with $S \subset \mathcal{B} \times \mathcal{B}$ and so on.

Definition 3.15. A **formal sum** (on a set \mathcal{V} of **values**) is a map $m : \mathcal{V} \rightarrow \mathbb{Z}$ that is zero outside a finite subset of \mathcal{V} .

Notation 3.16. We write $m : \mathcal{V} \rightarrow \mathbb{Z}$ as $\sum m(v)[v]$, perhaps omitting terms where $m(v) = 0$.

Conversely, if $h : \mathcal{B} \rightarrow \mathcal{V}$ is a Galois invariant and $R \in \mathcal{B}$ then

$$S_R = \{Y | h(Y) = h(R)\} \subset \mathcal{B} \quad (2)$$

is also Galois invariant, and so can be used as in the previous paragraph.

In this way, by alternating Galois invariant maps $\mathcal{B} \rightarrow \mathcal{V}$ as in (1), and finite subsets $S \subset \mathcal{B}$ as in (2), we can construct a tower of Galois invariants. For completeness, this process should

be extended to include $\mathcal{B} \times \mathcal{B}$ and so on. The process produces formal sums of formal sums and so on. One wants as many invariants as possible, while at the same time managing the duplication and redundancy that results. These matters will be further discussed in [4].

3.7. Covering spaces and $\pi_1(\hat{X})$. From now until the end of this section we will rely on some concepts and results from topology, which we will use to motivate the definition of the decorated lattice \mathcal{L} and to outline the proof of its Galois invariance. This results in many new invariants, to which the just described tower construction can be applied. What follows is intended for experts in dessins. Others may find it hard.

A map $f : Y \rightarrow X$ of topological spaces is a *covering map* if $f^{-1}(U)$ is the disjoint union of copies of U , for small enough open subsets U of X . The Shabat condition ensures that $f : \mathbb{C} \rightarrow \mathbb{C}$ is a covering map away from v_b and v_w .

The *fundamental group* $\pi_1(X, x_0)$ consists of all continuous maps $p : [0, 1] \rightarrow X$ with $p(0) = p(1) = x_0$, considered up to homotopy equivalence. Following first path p and then path q gives the group law on $\pi_1(X, x_0)$. This definition relies on the choice of a base point x_0 (and each path from x_0 to x_1 induces an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$). The subgroups of $\pi_1(X, x_0)$ are related to the covers of X .

A *pointed topological space* \hat{X} is a topological space X together with a base point x_0 . We let $\pi_1(\hat{X})$ denote $\pi_1(X, x_0)$. Suppose $f : Y \rightarrow X$ is a covering map, with $f(y_0) = x_0$. Write \hat{Y} for the pointed topological space (Y, y_0) and similarly for \hat{X} . We will say that $f : \hat{Y} \rightarrow \hat{X}$ is a *pointed covering map*.

Theorem 3.17. *Provided \hat{X} is connected and locally path connected, the connected pointed covers $f : \hat{Y} \rightarrow \hat{X}$ correspond to the subgroups $\pi_1(\hat{X})$, and vice versa.*

This theorem applies in our situation, with $X = \mathbb{C} \setminus \{-1, +1\}$ and $x_0 = 0$. Each biased dessin R produces a finite pointed cover $\hat{Y}_R \rightarrow \hat{X}$. The relation $R \rightarrow S$ on biased dessins, translated to topology, is equivalent to: The pointed covers $\hat{Y}_R \rightarrow \hat{X}$ and $\hat{Y}_S \rightarrow \hat{X}$ are such that (i) there is a pointed cover map $\hat{Y}_R \rightarrow \hat{Y}_S$, and (ii) the composite $\hat{Y}_R \rightarrow \hat{Y}_S \rightarrow \hat{X}$ is $\hat{Y}_R \rightarrow \hat{X}$.

From this, and standard results that produce a Belyi pair from a finite cover of X , it follows that the relation $R \rightarrow S$ on biased dessin (and hence the lattice structure) is Galois invariant (Theorem 3.14). Biased dessins (and maps between them) correspond to finite pointed covers of $\mathbb{C} \setminus \{-1, +1\}$ (and maps between them).

3.8. $\pi_1(\hat{X})$ and the lattice structure. By design, each Shabat polynomial gives a covering space (away from v_b and v_w), with a finite number of sheets. The same goes for Belyi pairs and $\mathbb{P}_1(\mathbb{C})$ less three points. Therefore, once bias has provided base points, we can apply Theorem 3.17.

Suppose H_R and H_S are subgroups of $G = \pi_1(\hat{X})$. In this situation both $H_R \cap H_S$ and $\langle H_R, H_S \rangle$ (the subgroup generated by H_R and H_S) are subgroups of G . This puts an order lattice structure on the subgroups of G . The construction of the join $R \vee S$ of two biased dessins (see Definition 3.6) corresponds to $H_R \cap H_S$ in $\pi_1(X, x_0)$, where $X = \mathbb{C} \setminus \{-1, +1\}$ and $x_0 = 0 \in X$.

3.9. The partition triple. We have just, via covering spaces, outlined why the lattice structure on \mathcal{B} is Galois invariant. This uses the global structure of biased dessins R and S to define

the relation $R \rightarrow S$. If we have $R \rightarrow S$ then there is also significant local structure that is Galois invariant. We will now outline how this produces from \mathcal{B} the decorated lattice \mathcal{L} .

Recall that R has permutations α_R and β_R acting on the edges E_R of R . Recall also that each black vertex of R is an α_R orbit in E_R . Thus, α partitions E_R into orbits, and hence produces a partition $p_{R,\alpha}$ of $n = \deg R$. We can similarly define $p_{R,\beta}$ and $p_{R,\gamma}$, where $\gamma = (\alpha\beta)^{-1}$ gives what is called the monodromy around $\infty \in \mathbb{P}_1(\mathbb{C})$. The following is easy and already known.

Proposition 3.18. *The partition triple $(p_{R,\alpha}, p_{R,\beta}, p_{R,\gamma})$ is a Galois invariant of $R \in \mathcal{B}$.*

The decoration that gives \mathcal{L} is a relative form of the partition triple. First a review. Let D_1 be the unique single-edged biased dessin. Given $R \rightarrow D_1$ we have marked Belyi pair $M_R \rightarrow \mathbb{P}_1(\mathbb{C})$. Further, the partition $p_{R,\alpha}$ gives Galois invariant information about the monodromy of $M_R \rightarrow \mathbb{P}_1(\mathbb{C})$ around $-1 \in \mathbb{P}_1(\mathbb{C})$, and similarly for $p_{R,\beta}$ and $p_{R,\gamma}$ around $+1$ and ∞ respectively.

Now suppose we have $R \rightarrow S \rightarrow D_1$. Each say black vertex v_r of R maps to a black vertex v_s of S (then to the black vertex of v_b of D_1 , which is what gives v_r and v_s their colour). Each vertex v_r of R has a multiplicity $\text{mult } v_r$ (number of edges that meet v_r). The numbers $\text{mult } v_r$, for all v_r mapping to v_b , give the partition $p_{R,\alpha}$.

The vertex v_r also maps to a vertex v_s on S . This gives additional information to record.

3.10. Decorating the lattice. Let \mathcal{L}' be \mathcal{B} considered as an abstract lattice, whose elements we will call *nodes*. Each node R is secretly a biased dessin, but for Galois purposes we are not allowed to look inside R and see the biased dessin. The underlying biased dessin is without Galois significance, which is why we keep it secret. However, some information does emerge.

The *decoration* of \mathcal{L}' consists of: (1) For each node R of \mathcal{L}' a finite set V_R , called the *vertices* of R . (2) A map $\text{mult} : V_R \rightarrow \mathbb{N}^+ = \{n > 0\}$. (3) Whenever $R \rightarrow S$, which now means the abstract partial order on \mathcal{L}' , there is a map $V_R \rightarrow V_S$.

Definition 3.19. *The decorated lattice of biased dessins \mathcal{L} is \mathcal{L}' decorated as above.*

We consider two decorations of a lattice to be equal if they are the same after relabelling, or in other words are related by bijections on the vertex sets V_R . Our decoration of \mathcal{L}' has special properties, such as (i) the maps $V_R \rightarrow V_S$ commute, and (ii) if $v_r \mapsto v_s$ under $V_R \rightarrow V_S$ then $\text{mult } v_r$ divides $\text{mult } v_s$. We don't need these properties in this paper. But we do care about automorphisms.

Definition 3.20. *An automorphism ψ of \mathcal{L} consists of a lattice isomorphism $\psi : \mathcal{L}' \rightarrow \mathcal{L}'$, together with maps $\psi : V_R \rightarrow V_{\psi(R)}$, such that (i) the composition $V_R \rightarrow V_{\psi(R)} \xrightarrow{\text{mult}} \mathbb{N}^+$ is equal to $V_R \xrightarrow{\text{mult}} \mathbb{N}^+$, and (ii) if $R \rightarrow S$ then the compositions $V_R \rightarrow V_{\psi(R)} \rightarrow V_{\psi(S)}$ and $V_R \rightarrow V_S \rightarrow V_{\psi(S)}$ are equal.*

Recall that \mathcal{L}' is an abstract lattice, each of whose nodes has secretly associated with it a biased dessin. Suppose ψ is automorphism of \mathcal{L} and R is a node of \mathcal{L} . Let U and $\psi(U)$ be the biased dessins secretly associated with R and $\psi(R)$. It is not required that ψ induce a bijection between the edges of U and those of $\psi(U)$. Recall that only two elements of Γ act continuously on $\overline{\mathbb{Q}} \subset \mathbb{C}$ (see §1.3). This might make it impossible to construct a bijection on the edges.

What ψ must do is preserve certain geometric relations between elements of \mathcal{B} . The lattice isomorphism $\psi : \mathcal{L}' \rightarrow \mathcal{L}'$ comes from global properties. The V_R , $\text{mult} : V_R \rightarrow \mathbb{N}^+$ and $V_R \rightarrow V_S$ come from local geometric properties.

Notation 3.21. $\Gamma' = \text{Aut}(\mathcal{L})$, the automorphism group of \mathcal{L} .

The bottom element D_1 of \mathcal{L} has three vertices, which we denote by v_b , v_w and v_∞ . Each has multiplicity one. Given a node R of \mathcal{L} , the map $V_R \rightarrow V_{D_1} = \{v_b, v_w, v_\infty\}$ partitions V_R into black, white and at-infinity vertices. The map $V_R \rightarrow \mathbb{N}^+$, restricted to each of these subsets, then gives the partition triple.

Each permutation of v_b, v_w, v_∞ induces an automorphism of \mathcal{L} . The following, given Theorem 2.19, is not hard. Its proof will be given in [3].

Notation 3.22. Γ'_0 is the subgroup of Γ' that fixes V_{D_1} .

Theorem 3.23. The absolute Galois group Γ is a subgroup of Γ'_0 .

At present, there is not evidence or a proof strategy for:

Conjecture 3.24. $\Gamma = \Gamma'_0$.

4 Conclusion

4.1. Summary. We have seen that adding bias to dessins brings many benefits. (1) Galois invariants can be defined directly from biased Shabat polynomials, say via minimal polynomials. (2) Biased dessins have a Galois invariant lattice structure, which can be used to help build a tower of Galois invariants. (3) Biased plane trees are counted by the Catalan numbers, which brings connections to many other parts of mathematics. (4) The decorated lattice \mathcal{L} of biased dessins is the ground for the definition of new Galois invariants, which generalise the partition triple. (5) The simply defined subgroup Γ'_0 of $\text{Aut}(\mathcal{L})$ contains, and might equal, the absolute Galois group Γ .

To this list we add: (6) Each $\psi \in \Gamma'_0$ induces a bijection $\psi : \mathcal{A} \rightarrow \mathcal{A}$, where $\mathcal{A} \subset \overline{\mathbb{Q}}$ are the coefficients that appear in \mathcal{S} . (7) We have additional structures and conjectures that can be explored using computer calculations. The purely combinatorial calculations might be easier.

Benefit (6) is important because $\psi \in \Gamma'_0$ will induce, and hence come from, a $\psi \in \Gamma$ just in case $\psi : \mathcal{A} \rightarrow \mathcal{A}$ respects all algebraic relations that exist between the elements of \mathcal{A} . This makes \mathcal{A} a potentially interesting object of study.

4.2. Two cultures. The minimal polynomial and the partition triple are both Galois invariants, but very different in character. The one is algebraic, the other combinatorial. They also apply to different types of object, namely elements of $\overline{\mathbb{Q}}$ and \mathcal{B} respectively. Thus, each type of object has its own type of Galois invariant.

The introduction of bias destroys this dichotomy. Each biased Shabat polynomial is, via the bridge, a biased plane tree and vice versa. As a biased Shabat polynomial it has ‘minimal polynomial’ style invariants. As a plane tree it has ‘partition triple’ style invariants.

Suppose we have a complete set \mathcal{X} of Galois invariants on, say, the algebraic number side. This means that *any* Galois invariant on the dessins side can be expressed using the \mathcal{X} invariants. The bridge will become more useful if we can produce sets of invariants \mathcal{X} and \mathcal{Y} , one at each end the bridge, that are *aligned*. By this I mean, for example, that $\mathcal{X}(f)$ and $\mathcal{Y}(T_f)$ are linear functions of each other. The author hopes to discuss this further in [4].

4.3. Unbiased Galois invariants. We have seen that biased Shabat polynomials and plane trees have many Galois invariants, coming from the coefficients of f and the lattice structure on \mathcal{B} respectively. Suppose, however, that our situation requires the study of unbiased objects. What now?

Formal sums allow Galois invariants to descend, solving this problem.

Proposition 4.1. *If h is a biased Galois invariant then*

$$h_{\Sigma}(X) = \sum_{Y'=X} [h(Y)]$$

is an unbiased Galois invariant. Here Y' means Y without its bias.

Proof. The set $S_X = \{Y \in \mathcal{B} | Y' = X\}$ is finite, and Galois covariant. □

This process can be thought of as *summing over the the bias* or *integrating over the fibre*.

4.4. Closing remarks. We have just seen how biased dessins naturally arise in the study unbiased dessins. We give the last word to Alexander Grothendieck, who seems to have anticipated this (see [6], p5 of AG's manuscript):

[L]e gens s'obstinent encore, en calculant avec des groupes fondamentaux, à fixer un seul point base, plutôt que d'en choisir astucieusement tout un paquet qui soit invariant par les symétries de la situation [...]

Or in English [7]:

[P]eople still obstinately persist, when calculating with fundamental groups, in fixing a single base point, instead of cleverly choosing a whole packet of points which is invariant under the symmetries of the situation [...]

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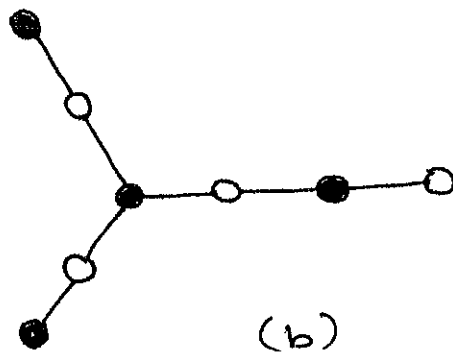
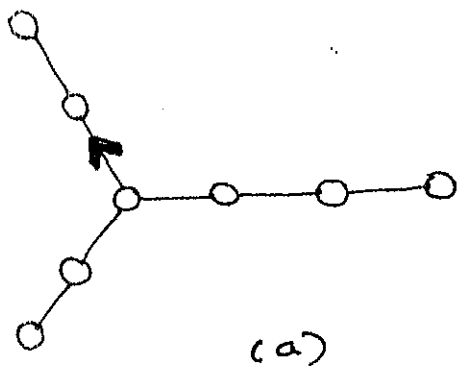


FIGURE 1.

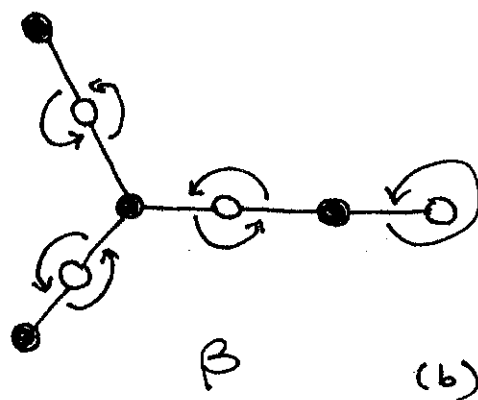
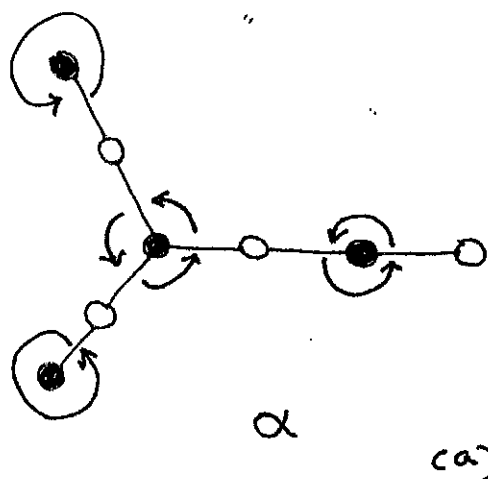


FIGURE 2

